THE CHINESE UNIVERSITY OF HONG KONG

Department of Mathematics

SAYT 1134 & Toward Differential Geometry

Tutorial 8

1 Eigenvector, eigenvalues and diagonalization

Steps for diagonalizing a $n \times n$ matrix A:

Step 1: Solve the equation $det(A - \lambda I) = 0$.

Step 2: For each λ found, find a vector $\mathbf{x}$ satisfying $A\mathbf{x} = \lambda\mathbf{x}$. (Note that in some cases, the number of linearly independent eigenvectors found can be more than 1.)

Step 3: Count the total number of linearly independent eigenvectors found for all eigenvalues:

Case 1: Total number of linearly independent eigenvectors found for all eigenvalues $< n$: The matrix is not diagonalizable.

Case 2: Total number of linearly independent eigenvectors found for all eigenvalues$= n$: The matrix is diagonalizable: Take $Q = [\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}]$, where $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}$ are linearly independent eigenvectors found. Then $D = Q^{-1}AQ$ is a diagonal matrix. Note that for $i = 1, 2, ..., n$, $[D]_{ii} = \lambda_i$, where λ_i is the coresponding eigenvalue of $\mathbf{v_i}$.

Example 1.1. *Find the eigenvalues and eigenvectors of the following matrices, and determine whether it is diagonalizable. If yes, find the diagonal matrix formed.*

i $\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$

ii $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

iii $\begin{pmatrix} 2 & 0 \\ 5 & 8 \end{pmatrix}$

iv $\begin{pmatrix} 4 & 2 \\ 2 & 0 \end{pmatrix}$

v $\begin{pmatrix} -1 & -2 & 2 \\ 4 & 3 & -4 \\ 0 & -2 & 1 \end{pmatrix}$

Besides, there are some interesting properties of diagonalization $D = Q^{-1}AQ$:

i $det(D) = det(Q^{-1}AQ) = det(Q)^{-1}det(A)det(Q) = det(A)$.

ii $tr(A) = tr(Q)$*: This is due to the property* $tr(CD) = tr(DC)$ *for all* $n \times n$ *matrices* C, D.

2 Normal curvature

Example 2.1. *Show that the sum of the normal curvatures for any pair of orthogonal directions at a point* $\mathbf{p} \in S$ *is a constant.*

Proof. Let κ_1, κ_2 be the associated principal curvatures at p respectively.
Note that from lecture notes, for all $\mathbf{v} \in \mathbf{T_p S}$ we have $\kappa_n(\mathbf{v}) = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta$ for some $\theta \in [0, 2\pi)$.

Let $\mathbf{v_1}$ and $\mathbf{v_2}$ be any pair of orthogonal directions at the point $\mathbf{p}$. Then without loss of generality, we let

$$\begin{aligned} \kappa(\mathbf{v_1}) &= \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta \\ \kappa(\mathbf{v_2}) &= \kappa_1 \cos^2(\theta + \frac{\pi}{2}) + \kappa_2 \sin^2(\theta + \frac{\pi}{2}) \\ &= \kappa_1 \sin^2\theta + \kappa_2 \cos^2\theta \end{aligned}$$

Then we have

$$\begin{aligned} \kappa(\mathbf{v_1}) + \kappa(\mathbf{v_2}) &= \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta + \kappa_1 \sin^2\theta + \kappa_2 \cos^2\theta \\ &= \kappa_1[\cos^2\theta + \sin^2\theta] + \kappa_2[\cos^2\theta + \sin^2\theta] \\ &= \kappa_1 + \kappa_2, \text{ which is a constant} \end{aligned}$$

□

3 Mean curvature

Definition 3.1 (Second fundamental form)**.**

$$II = \begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} \langle \mathbf{x}_{uu}, \mathbf{n} \rangle & \langle \mathbf{x}_{uv}, \mathbf{n} \rangle \\ \langle \mathbf{x}_{vu}, \mathbf{n} \rangle & \langle \mathbf{x}_{vv}, \mathbf{n} \rangle \end{pmatrix} = - \begin{pmatrix} \langle \mathbf{x}_u, \mathbf{n}_u \rangle & \langle \mathbf{x}_u, \mathbf{n}_v \rangle \\ \langle \mathbf{x}_v, \mathbf{n}_u \rangle & \langle \mathbf{x}_v, \mathbf{n}_v \rangle \end{pmatrix}.$$

Definition 3.2 (Mean curvature)**.** *Let* S *be a regular surface and* $d\mathbf{n}_p$ *be the differential of Gauss map at* $p \in S$*. The* **mean curvature** *of* S *at* p *is*

$$H = -\frac{1}{2} tr(d\mathbf{n}_p) = \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{1}{2} tr((II)(I^{-1})) = \frac{1}{2}\left(\frac{gE - 2fF + eG}{EG - F^2}\right).$$

where $tr(d\mathbf{n}_p)$ *is the trace of* $d\mathbf{n}_p$ *and* κ_1, κ_2 *are the principal curvatures of* S *at* p*.*

Example 3.1. *Find the mean curvature of the following parametrized surface:* $\mathbf{x}(u, v) = (u, v, uv)$*,* $u, v \in \mathbb{R}$*.*

Solution. *Note that*

$$\begin{aligned}
X_u &= (1,0,v)\\
X_v &= (0,1,u)\\
X_u \times X_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 0 & v\\ 0 & 1 & u \end{vmatrix}\\
&= -v\mathbf{i} - u\mathbf{j} + \mathbf{k}\\
\mathbf{n} &= \frac{X_u \times X_v}{\|X_u \times X_v\|}\\
&= \frac{(-v,-u,1)}{\sqrt{1+u^2+v^2}}\\
X_{uu} &= (0,0,0)\\
X_{uv} &= (0,0,1)\\
X_{vv} &= (0,0,0)\\
I &= \begin{pmatrix} \langle \mathbf{x}_u, \mathbf{x}_u\rangle & \langle \mathbf{x}_u, \mathbf{x}_v\rangle\\ \langle \mathbf{x}_v, \mathbf{x}_u\rangle & \langle \mathbf{x}_v, \mathbf{x}_v\rangle \end{pmatrix} = \begin{pmatrix} 1+v^2 & uv\\ uv & 1+u^2 \end{pmatrix}\\
II &= \begin{pmatrix} \langle \mathbf{x}_{uu}, \mathbf{n}\rangle & \langle \mathbf{x}_{uv}, \mathbf{n}\rangle\\ \langle \mathbf{x}_{vu}, \mathbf{n}\rangle & \langle \mathbf{x}_{vv}, \mathbf{n}\rangle \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{\sqrt{1+u^2+v^2}}\\ \frac{1}{\sqrt{1+u^2+v^2}} & 0 \end{pmatrix}
\end{aligned}$$

(Of course we don't want to compute $\mathbf{n_u}$ *and* $\mathbf{n_v}$*,right?)*

So we have

$$\begin{aligned}
H &= \frac{1}{2}\left(\frac{gE-2fF+eG}{EG-F^2}\right)\\
&= \frac{1}{2}\left(\frac{-2fF}{EG-F^2}\right)\\
&= -\frac{\frac{uv}{(1+u^2+v^2)^{-\frac{1}{2}}}}{1+u^2+v^2}\\
&= -\frac{uv}{(1+u^2+v^2)^{\frac{3}{2}}}
\end{aligned}$$

Example 3.2 (Isothermal coordinate)**.** *Show that for a regular surface* $X(u,v)$ *with normal* $\mathbf{n} = \dfrac{X_u \times X_v}{\|X_u \times X_v\|}$*, if* $I = \begin{pmatrix} f^2 & 0\\ 0 & f^2 \end{pmatrix}$ *for some smooth function* f*, then*

$$X_{uu} + X_{vv} = 2f^2 H\mathbf{n}.$$

This coordinate system is called the ***isothermal coordinate****.*

Solution. *The statement shows that* $X_{uu} + X_{vv}$ *is parallel to* $2f^2H\mathbf{n}$*. So apart from checking that the coefficient of* $\mathbf{n}$ *is* $2f^2H$*, we also need to check that* $X_{uu} + X_{vv}$ *is*

independent of "influence" *of other vectors that are linearly independent of* $\mathbf{n}$*. To begin with, let's find a basis for* $\mathbb{R}^3$ *containing* $\mathbf{n}$*:*
Note that by regularity, we have X_u *and* X_v *are linearly independent non-zero vectors, hence* $X_u \times X_v \neq \mathbf{0}$*.*
Besides, by definition of $\mathbf{n}$*, we have* $\mathbf{n} \perp X_u, X_v$*.*
Hence $\{X_u, X_v, \mathbf{n}\}$ *forms a basis for* $\mathbb{R}^3$*.*
Let

$$X_{uu} + X_{vv} = \alpha X_u + \beta X_v + \zeta \mathbf{n}.$$

We need to show that

i $\zeta = 2f^2 H$.

ii $\alpha = \beta = 0$.

So let's show these two steps separately:

i *Note that*

$$\begin{aligned} H &= \frac{1}{2}\left(\frac{gE - 2fF + eG}{EG - F^2}\right) \\ &= \frac{1}{2}\left(\frac{gf^2 + ef^2}{f^4}\right) \\ &= \frac{e+g}{2f^2} \end{aligned}$$

Taking inner product with $\mathbf{n}$ *on both sides (so we can eliminate* α, β *by* $< X_u, \mathbf{n} >=< X_v, \mathbf{n} >= 0$*), we have:*

$$\begin{aligned} < X_{uu} + X_{vv}, \mathbf{n} > &= \alpha < X_u, \mathbf{n} > + \beta < X_v, \mathbf{n} > + \zeta < \mathbf{n}, \mathbf{n} > \\ < X_{uu}, \mathbf{n} > + < X_{vv}, \mathbf{n} > &= \zeta(1) \text{(As } \mathbf{n} \text{ has norm 1.)} \\ e + g &= \zeta \\ \zeta &= 2f^2 H \end{aligned}$$

ii *Taking inner product with* X_u *on both sides (so we can eliminate* β *by* $< X_u, \mathbf{n} > 0$*), we have:*

$$\begin{aligned} < X_{uu} + X_{vv}, X_u > &= \alpha < X_u, X_u > + \beta < X_v, X_u > + \zeta < \mathbf{n}, X_u > \\ < X_{uu}, X_u > + < X_{vv}, X_u > &= \alpha(f^2) + \beta(0) \\ \alpha &= \frac{1}{f^2}[< X_{uu}, X_u > + < X_{vv}, X_u >] \end{aligned}$$

So we need to show that

$$< X_{uu}, X_u > + < X_{vv}, X_u >= 0 :$$

Thinking process:

(1) *How to make* $< X_{uu}, X_u >$*?*
Note that $< X_{uu}, X_u > = < (X_u)_u, X_u >$.
Recall we have the following technique : $< x, y >' = < x', y > + < x, y' >$.
In particular, we have $< x, x >' = 2 < x, x' >$.
Then we have $< X_{uu}, X_u > = < (X_u)_u, X_u > = \frac{1}{2}\partial_u < X_u, X_u > = \frac{1}{2}\partial_u(f^2) = ff_u$.

(2) *How to make* $< X_{vv}, X_u >$*?*
Note that $< X_{vv}, X_u > = < (X_v)_v, X_u >$.
Recall that $< X_u, X_v > = 0$ *as given in the question.*
Then note that

$$\begin{aligned}
< X_u, X_v > &= 0 \\
\partial_v < X_u, X_v > &= 0 \\
< X_{vv}, X_u > + < X_{uv}, X_v > &= 0 \\
< X_{vv}, X_u > + < (X_v)_u, X_v > &= 0 \\
< X_{vv}, X_u > + \frac{1}{2}\partial_u < X_v, X_v > &= 0 \\
< X_{vv}, X_u > + \frac{1}{2}\partial_u(f^2) &= 0 \\
< X_{vv}, X_u > + 2ff_u &= 0 \\
< X_{vv}, X_u > &= -ff_u
\end{aligned}$$

Hence we have

$$< X_{uu}, X_u > + < X_{vv}, X_u > = ff_u - ff_u = 0.$$

Similarly, for $< X_{uu}, X_v > + < X_{vv}, X_v >$*, we have*

$$\begin{aligned}
< X_{vv}, X_v > &= \frac{1}{2}\partial_v < X_v, X_v > \\
&= ff_v
\end{aligned}$$

and

$$\begin{aligned}
\partial_u < X_u, X_v > &= 0 \\
< X_{uu}, X_v > + < X_u, X_{vu} > &= 0 \\
< X_{uu}, X_v > &= -\frac{1}{2}\partial_v < X_u, X_u > \\
&= -ff_v
\end{aligned}$$

Hence we have

$$< X_{uu}, X_v > + < X_{vv}, X_v > = ff_v - ff_v = 0.$$

Hence we have

$$X_{uu} + X_{vv} = 2f^2H\mathbf{n}.$$

Remark. Therefore, an easy corollary is that the surface is **minimal if and only if** $\mathbf{X_{uu}} + \mathbf{X_{vv}} = \mathbf{0}$ (This type of function is called **harmonic**, and $\mathbf{X_{uu}} + \mathbf{X_{vv}}$ is called the **Laplacian** of X, denoted by ΔX.)

Example 3.3. *Using the result of the previous exercise, show that the helicoid:*

$$X(u,v) = (a \sinh v \cos u, a \sinh v \sin u, au)$$

and the Enneper's surface

$$X(u,v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2\right)$$

are minimal surfaces.

Solution. *Helicoid:*

$$\begin{aligned}
X_u &= (-a \sinh v \sin u, a \sinh v \cos u, a) \\
X_v &= (a \cosh v \cos u, a \cosh v \sin u, 0) \\
X_{uu} &= (-a \sinh v \cos u, -a \sinh v \sin u, 0) \\
X_{vv} &= (a \sinh v \cos u, a \sinh v \sin u, 0) \\
E &= < X_u, X_u > \\
&= a^2 \sinh^2 v[\sin^2 u + \cos^2 u] + a^2 \\
&= a^2 \sinh^2 v + a^2 \\
&= a^2[\sinh^2 v + 1] \\
&= a^2 \cosh^2 v \ \textit{(By } \cosh^2 v - \sinh^2 v = 1 \textit{ and hence } \sinh^2 v + 1 = \cosh^2 v\textit{)} \\
F &= < X_u, X_v > \\
&= -a^2 \sinh v \cosh v \sin u \cos u + a^2 \sinh v \cosh v \sin u \cos u \\
&= 0 \\
G &= < X_v, X_v > \\
&= a^2 \cosh^2 v[\cos^2 u + \sin^2 u] \\
&= a^2 \cosh^2 v
\end{aligned}$$

So we have $E = G = (a \cosh v)^2$ and $F = 0$.
Also, we have $X_{uu} + X_{vv} = 0$.
Thereby, it is an isothermal coordinate and thereby $H = 0$.
So the surface is minimal.

Enneper's surface:

$$
\begin{aligned}
X_u &= \big(1-u^2+v^2, 2uv, 2u\big)\\
X_v &= \big(2uv, 1-v^2+u^2, -2v\big)\\
X_{uu} &= (-2u, 2v, 2)\\
X_{vv} &= (2u, -2v, -2)\\
E &=< X_u, X_u >\\
&= (1-u^2+v^2)^2 + 4u^2v^2 + 4u^2\\
&= 1 + u^4 + v^4 - 2u^2 + 2v^2 - 2u^2v^2 + 4u^2v^2 + 4u^2\\
&= u^4 + v^4 + 2u^2 + 2v^2 + 2u^2v^2 + 1\\
&= (1+u^2+v^2)^2\\
F &=< X_u, X_v >\\
&= 2uv - 2u^3v + 2uv^3 + 2uv - 2uv^3 + 2u^3v - 4uv\\
&= 0\\
G &=< X_v, X_v >\\
&= 4u^2v^2 + (1-v^2+u^2)^2 + 4v^2\\
&= 4u^2v^2 + 1 + v^4 + u^4 - 2v^2 + 2u^2 - 2u^2v^2 + 4v^2\\
&= u^4 + v^4 + 2u^2 + 2v^2 + 2u^2v^2 + 1\\
&= (1+u^2+v^2)^2
\end{aligned}
$$

Remark. We have

$$
\begin{aligned}
(a+b+c)^2 &= a^2+b^2+c^2+2ab+2ac+2bc\\
(a+b-c)^2 &= a^2+b^2+c^2+2ab-2ac-2bc
\end{aligned}
$$

Hence we have $E = G = (1+u^2+v^2)^2$ and $F = 0$.
Also, we have $X_{uu} + X_{vv} = 0$.
Thereby, it is an isothermal coordinate and thereby $H = 0$.
So the surface is minimal.

Exercise 3.1. Prove that the surface defined by $z = f(x,y)$ is a minimal surface if and only if

$$(1+f_x^2)f_{yy} - 2f_xf_yf_{xy} + (1+f_y^2)f_{xx} = 0.$$

Solution. Note that $F(x,y) = (x, y, f(x,y))$ is a parametrization of the surface.
Then we have

$$
\begin{aligned}
F_x &= (1, 0, f_x) \\
F_y &= (0, 1, f_y) \\
I &= \begin{pmatrix} 1+f_x^2 & f_x f_y \\ f_x f_y & 1+f_y^2 \end{pmatrix} \\
F_x \times F_y &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} \\
&= (-f_x, -f_y, 1) \\
\mathbf{n} &= \frac{F_x \times F_y}{\|F_x \times F_y\|} \\
&= \frac{(-f_x, -f_y, 1)}{\sqrt{1+f_x^2+f_y^2}} \\
F_{xx} &= (0, 0, f_{xx}) \\
F_{yy} &= (0, 0, f_{yy}) \\
F_{xy} &= (0, 0, f_{xy}) \\
II &= \begin{pmatrix} \langle \mathbf{x}_{uu}, \mathbf{n} \rangle & \langle \mathbf{x}_{uv}, \mathbf{n} \rangle \\ \langle \mathbf{x}_{vu}, \mathbf{n} \rangle & \langle \mathbf{x}_{vv}, \mathbf{n} \rangle \end{pmatrix} = \begin{pmatrix} \dfrac{f_{xx}}{\sqrt{1+f_x^2+f_y^2}} & \dfrac{f_{xy}}{\sqrt{1+f_x^2+f_y^2}} \\ \dfrac{f_{xy}}{\sqrt{1+f_x^2+f_y^2}} & \dfrac{f_{yy}}{\sqrt{1+f_x^2+f_y^2}} \end{pmatrix}
\end{aligned}
$$

(Of course we don't want to compute $\mathbf{n_u}$ and $\mathbf{n_v}$,right?)

Therefore, we have

$$
\begin{aligned}
H &= \frac{1}{2}\left(\frac{gE - 2fF + eG}{EG - F^2}\right) \\
&= \frac{1}{2} \times \frac{1}{(1+f_x^2)(1+f_y^2) - f_x^2 f_y^2} \times \frac{f_{yy}(1+f_x^2) - 2f_{xy}f_x f_y + f_{xx}(1+f_y^2)}{\sqrt{1+f_x^2+f_y^2}}
\end{aligned}
$$

Note that $H = 0$ if and only if the numerator$= 0$.
Therefore, we have $H = 0$ if and only if

$$f_{yy}(1+f_x^2) - 2f_{xy}f_x f_y + f_{xx}(1+f_y^2) = 0.$$

Exercise 3.2.

i Find the mean curvature of Helicoid: $\mathbf{x}(u, \theta) = (au\cos\theta, au\sin\theta, b\theta)$, $u, \theta \in \mathbb{R}$ where $a, b > 0$ are constants.

ii Given a regular surface $S \subset \mathbb{R}^3$ with orientation $\mathbf{N}$. Let $\mathbf{p} \in S$, show that the mean curvature H at $\mathbf{p} \in S$ is given by

$$H = \frac{1}{\pi}\int_0^\pi k_n(\theta)\ d\theta,$$

where $k_n(\theta)$ is the normal curvature at $\mathbf{p}$ along the direction making an angle θ with a fixed direction.

Solution.

i *Note that*

$$\begin{aligned}
X_u &= (a\cos\theta, a\sin\theta, 0)\\
X_\theta &= (-au\sin\theta, au\cos\theta, b)\\
X_u \times X_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ a\cos\theta & a\sin\theta & 0\\ -au\sin\theta & au\cos\theta & b \end{vmatrix}\\
&= ab\sin\theta\mathbf{i} - ab\cos\theta\mathbf{j} + [a^2u\cos^2\theta + a^2u\sin^2\theta]\mathbf{k}\\
&= ab\sin\theta\mathbf{i} - ab\cos\theta\mathbf{j} + a^2u\mathbf{k}\\
\mathbf{n} &= \frac{X_u \times X_v}{\|X_u \times X_v\|}\\
&= \frac{(ab\sin\theta, -ab\cos\theta, a^2u)}{\sqrt{a^2b^2 + a^4u^2}}\\
&= \frac{(b\sin\theta, -b\cos\theta, au)}{\sqrt{b^2 + a^2u^2}}\\
X_{uu} &= (0,0,0)\\
X_{u\theta} &= (-a\sin\theta, a\cos\theta, 0)\\
X_{\theta^2} &= (-au\cos\theta, -au\sin\theta, 0)\\
X_{u\theta} &= (-a\sin\theta, a\cos\theta, 0)\\
I &= \begin{pmatrix} a^2 & 0\\ 0 & a^2u^2 + b^2 \end{pmatrix}\\
II &= \frac{1}{\sqrt{b^2 + a^2u^2}} \begin{pmatrix} 0 & -ab\sin^2\theta - ab\cos^2\theta\\ -ab\sin^2\theta - ab\cos^2\theta & -abu\sin\theta\cos\theta + abu\sin\theta\cos\theta \end{pmatrix}\\
&= \frac{1}{\sqrt{b^2 + a^2u^2}} \begin{pmatrix} 0 & -ab\\ -ab & 0 \end{pmatrix}
\end{aligned}$$

So we have

$$\begin{aligned}
H &= \frac{1}{2}\left(\frac{gE - 2fF + eG}{EG - F^2}\right)\\
&= \frac{1}{2}\left(\frac{(0)E - 2f(0) + (0)G}{EG}\right)\\
&= 0
\end{aligned}$$

(Remark: Note that this surface ***does not have isothermal coordinate!****)*

ii *Note that from lecture notes, we have*

$$k_n(\theta) = k_1\cos^2\theta + k_2\sin^2\theta$$

, *where* κ_1, κ_2 *be the associated principal curvatures at* p *respectively. Then we have*

$$\begin{aligned}
\frac{1}{\pi}\int_0^\pi k_n(\theta)\ d\theta &= \frac{1}{\pi}\int_0^\pi k_1\cos^2\theta + k_2\sin^2\theta\ d\theta \\
&= \frac{1}{\pi}\int_0^\pi k_1\cdot\frac{1+\cos 2\theta}{2} + k_2\cdot\frac{1-\cos 2\theta}{2}\ d\theta \\
&= \frac{1}{2\pi}\left[k_1\theta + \frac{k_1\sin 2\theta}{2} + k_2\theta - \frac{k_2\sin 2\theta}{2}\right]_0^\pi \\
&= \frac{k_1\pi + k_2\pi}{2\pi} \\
&= \frac{k_1+k_2}{2} \\
&= H
\end{aligned}$$

4 Appendix: Why do we need to care the mean curvature?

(Extracted from 2023-2024 MATH 4030 Lecture notes)
(Please be assured that this part will definitely not appear in the exam. It it just for those who want to know why we need to care the mean curvature.)

Given $X : U \to M \subset \mathbb{R}^3$ ($M = X(U)$ is a regular parameterized surface.) Let D be a bounded open domain in U,
Consider $X^t : U \to M$ given by

$$X^t(\mathbf{p}) := X(\mathbf{p}) + th(\mathbf{p})\mathbf{n}(\mathbf{p})$$

, where $\mathbf{p} \in D$, $\mathbf{n}(\mathbf{p})$ is the normal of the surface at $X(\mathbf{p})$ and $h : \overline{D} \to \mathbb{R}$ be a smooth (i.e. infinitely many times differentable) function and for very small values of t.

$th(\mathbf{p})\mathbf{n}(\mathbf{p})$ is defined to be the **normal variation** of M. We want to find the **"rate of change"/derivative** of the **area** of $X(U)^t$, denote by A^t, under **little normal variation** from the original plane $X(U)$ (i.e. $t = 0$).
The appearance of derivative $\left.\frac{d}{dt}\right|_{t=0} A^t$ makes only considering small values of t to make sense.

Then we want to prove the following theorem:

Theorem 4.1. *With the notation above, we have*

$$\left.\frac{d}{dt}\right|_{t=0} Area(X^t(D)) = \int_D -2hH\sqrt{detg}\ dudv.$$

Note that we have the following proposition:

Proposition 4.2. X^t *is a parametrization for very small values of* t *(Notation:* $|t| << 1$.*)*

Proof.

$$\begin{cases} X_u^t = X_u + t(h\mathbf{n})_u = X_u + th_u\mathbf{n} + th\mathbf{n}_u \\ X_v^t = X_v + t(h\mathbf{n})_v = X_v + th_v\mathbf{n} + th\mathbf{n}_v \end{cases}.$$

To show that a surface $X^t(u,v)$ is regular parameterized, we need to show that the tangent vectors X_u^t and X_v^t are always linearly independent (equivalent to $X_u^t \times X_v^t \neq \mathbf{0}$):

$$\begin{aligned} X_u^t \times X_v^t &= [X_u + th_u\mathbf{n} + th\mathbf{n}_u] \times [X_v + th_v\mathbf{n} + th\mathbf{n}_v] \\ &= X_u \times X_v + t[h_u\mathbf{n} \times X_v + h_v X_u \times \mathbf{n}] + t^2 h[h_u\mathbf{n} \times \mathbf{n}_v + h_v\mathbf{n} \times \mathbf{n}_u] \end{aligned}$$

It is true that for a continuous function on closed and bounded set, maximum and minimum value exists. (You can just treat it as fact first. You will learn how to prove it in a mathematical analysis course.)
As $\|h_u\mathbf{n} \times X_v + h_v X_u \times \mathbf{n}\|$ and $\|h_u\mathbf{n} \times \mathbf{n}_v + h_v\mathbf{n} \times \mathbf{n}_u\|$ are continuous (just believe it first), they are bounded. Then we can choose very small value of t such that $\|X_u \times X_v\| > \|t[h_u\mathbf{n} \times X_v + h_v X_u \times \mathbf{n}] + t^2 h[h_u\mathbf{n} \times \mathbf{n}_v + h_v\mathbf{n} \times \mathbf{n}_u]\|$, making $X_u^t \times X_v^t \neq \mathbf{0}$.
(Obviously, $\|\mathbf{a}\| \neq \|\mathbf{b}\|$ implies $\mathbf{a} \neq \mathbf{b}$.
In university courses, most of the time you just know the existence of a thing, but it is very unlikely for you to find what the value is.)

Hence the tangent vectors X_u^t and X_v^t are always linearly independent and therefore $M_t = X^t(u)$ is a regular surface for very small values of t.

□

To find the area of the surface, recall that

$$A = \int_D \sqrt{det(I)}\ dudv$$

, where $I = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} < X_u, X_u > & < X_u, X_v > \\ < X_v, X_u > & < X_v, X_v > \end{pmatrix}$ is the first fundamental form at $X(\mathbf{p})$.
Also, denote $II = \begin{pmatrix} e & f \\ f & g \end{pmatrix} = -\begin{pmatrix} < X_u, \mathbf{n}_u > & < X_u, \mathbf{n}_v > \\ < X_v, \mathbf{n}_u > & < X_v, \mathbf{n}_v > \end{pmatrix}$ is the second fundamental form at $X(\mathbf{p})$. Then denote $I^t = \begin{pmatrix} E^t & F^t \\ G^t & H^t \end{pmatrix}$ is the first fundamental form at $X^t(\mathbf{p})$.

Note that

$$\begin{aligned} F^t &= < X_u^t, X_v^t > \\ &= < X_u + th_u\mathbf{n} + th\mathbf{n}_u, X_v + th_v\mathbf{n} + th\mathbf{n}_v > \\ &= < X_u, X_v > + th[< X_u, \mathbf{n}_v > + < X_v, \mathbf{n}_u >] + t^2(h_u h_v + h^2 < \mathbf{n}_u, \mathbf{n}_v >) \\ &\quad (\text{As } X_u, X_v \perp \mathbf{n},\ < X_u, \mathbf{n} > = < X_v, \mathbf{n} > = 0.) \\ &\quad (\text{Also, as } < \mathbf{n}.\mathbf{n} > \equiv 1, \text{ we have } 0 = < \mathbf{n}.\mathbf{n} >_u = 2 < \mathbf{n}_u, \mathbf{n} > \text{ and } 0 = < \mathbf{n}.\mathbf{n} >_v = 2 < \mathbf{n}_v, \mathbf{n} >) \\ &= F - 2thf + t^2(h_u h_v + h^2 < \mathbf{n}_u, \mathbf{n}_v >) \\ \partial_t F^t|_{t=0} &= [-2hf + 2t(h_u h_v + h^2 < \mathbf{n}_u, \mathbf{n}_v >)]_{t=0} \\ &= -2hf \end{aligned}$$

Similarly, we have

$$\begin{aligned}
E^t &=< X_u^t, X_u^t > \\
&=< X_u + th_u\mathbf{n} + th\mathbf{n}_u, X_u + th_u\mathbf{n} + th\mathbf{n}_u > \\
&=< X_u, X_u > + th[< X_u, \mathbf{n}_u > + < X_u, \mathbf{n}_u >] + t^2(h_u^2 + h^2 < \mathbf{n}_u, \mathbf{n}_u >) \\
&= E - 2the + t^2(h_u^2 + h^2 < \mathbf{n}_u, \mathbf{n}_u >) \\
\partial_t E^t|_{t=0} &= [-2he + 2t(h_u^2 + h^2 < \mathbf{n}_u, \mathbf{n}_u >)]_{t=0} \\
&= -2he
\end{aligned}$$

and

$$\begin{aligned}
G^t &=< X_v^t, X_v^t > \\
&=< X_v + th_v\mathbf{n} + th\mathbf{n}_v, X_v + th_v\mathbf{n} + th\mathbf{n}_v > \\
&=< X_v, X_v > + th[< X_u, \mathbf{n}_v > + < X_v, \mathbf{n}_v >] + t^2(h_v^2 + h^2 < \mathbf{n}_v, \mathbf{n}_v >) \\
&= G - 2thg + t^2(h_u^2 + h^2 < \mathbf{n}_u, \mathbf{n}_u >) \\
\partial_t G^t|_{t=0} &= [-2hg + 2t(h_u^2 + h^2 < \mathbf{n}_u, \mathbf{n}_u >)]_{t=0} \\
&= -2hg
\end{aligned}$$

Therefore, we have

$$\partial_t I^t|_{t=0} = -2h \begin{pmatrix} e & f \\ f & g \end{pmatrix} = -2hII$$

Recall the theorem that we want to prove:

$$\frac{d}{dt} Area(X^t(D)) = \int_D -2hH\sqrt{detg}\ dudv.$$

Note that the area A^t of $X^t(D)$ $A^t = \int_D dA^t$. By fundamental theorem of calculus, integration and differentiation sign can excahnge, so the theorem can be transformed to

$$\int_D -2hH\sqrt{detg}\ dudv = \left.\frac{d}{dt}\right|_{t=0} \int_D dA^t = \int_D \left.\frac{d}{dt}\right|_{t=0} dA^t.$$

So let's go to find an expression of dA^t. By comparing both sides, we want to prove that

$$\left.\frac{d}{dt}\right|_{t=0} dA^t = -2hH\sqrt{detg}\ dudv$$

Also, we have

Proposition 4.3 (Jacabi formula). *For a matrix function $A(t)$ such that $A(t)$ is invertible for all small values of t, we have*

$$\frac{\partial}{\partial t} det(A) = det(A) tr(A^{-1}\frac{dA}{dt})$$

(Go to Wiki for the proof. It is computational exhaustive.)

Therefore, with I^t is invertible, we have the derivative

$$\begin{aligned}
\frac{d}{dt}\bigg|_{t=0} dA^t &= \frac{d}{dt}\bigg|_{t=0} \sqrt{det(I^t)}\ dudv \\
&= \frac{1}{2}\{\frac{1}{\sqrt{det(I^t)}}[det(I^t)]'\}_{t=0}\ dudv \\
&= \frac{1}{2\sqrt{det(I)}} det(I) tr(I^{-1}\ \frac{dI}{dt}\bigg|_{t=0})\ dudv \text{ (By Proposition 4.3)} \\
&= \frac{1}{2\sqrt{det(I)}} det(I) tr[I^{-1}(-2hII)]\ dudv \\
&= -h\sqrt{det(I)} tr[I^{-1}(-II)]\ dudv \\
&= -2hH\sqrt{det(I)}\ dudv \text{ (As } tr(AB) = tr(BA)\text{, hence } tr[I^{-1}(II)] = tr[(II)I^{-1}]. \\
&\text{so } tr[I^{-1}(II)] = \kappa_1 + \kappa_2 = 2H.)
\end{aligned}$$

So we gave

$$\frac{d}{dt} Area(X^t(D)) = \int_D -2hH\sqrt{detg}\ dudv.$$